Multiplication schema for signed number:
Case study of three prospective teachers

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Abstract

This study investigated the pedagogical content knowledge that a college learner who is a prospective teacher might construct for teaching two-factor multiplication. In particular in this report, we attended to learners' cognitive structures for signed number multiplication, described in terms of actions, processes, objects, and schema. In closing, we suggest problem-posing, visualization of problem solving, and identifying the isomorphic relationship between computation and visualization as tools for improving both future research and the college mathematics preparation of teachers.

Introduction

The exercise of problem-posing may serve as a vehicle to assess and to enhance students' ability to connect symbolic representations to real life situations (Pirie, 2002). In this study, we expanded on problem-posing to incorporate visualizing problem solving and identifying isomorphic relationship between visualization of problem solving and numerical computation to explore participants' structures of understanding. In this report on part of the study, the focus is on prospective teachers' working with multiplication of signed numbers. Though studies exist on children learning addition and subtraction of negative numbers (e.g., Peled, 1991), research regarding prospective teachers' understanding of multiplication with signed numbers is scarce. Meanwhile, studies looking at understanding of fraction multiplication through problem posing abound, most with school children and some with prospective teachers. In particular, Rizvi (2004) reported that, prior to specific instruction on the topic, no prospective teacher participants were able to pose appropriate world problems for division numerical prompts when divisors were fractions. The relationship between signed number and fraction multiplication is structural: in each case a bi-partite form (sign, number) or (numerator, denominator) is combined with another. In the case of signed number, the combination is bi-linear, in the case of fraction it is not.

Two-factor multiplication with signed number

The *multiplier* in a multiplication of two factors is the number of equivalent collections while the *multiplicand* is the size of each collection. In the United States, the conventional expression of multiplication is *multiplier* × *multiplicand* = *product*. For whole numbers, grouping of values occurs (the multiplicand) and grouping of groups occurs (action of the multiplier). When the multiplier is a signed integer, further efforts at having sign in the multiplier act on the sign in the multiplicand are needed.

Chip, charged-field, and number-line are models frequently used in college textbooks for prospective teachers to explain multiplication with signed numbers (e.g., Bennett & Nelson, 2001). In such texts, the properties of the action of the sign in the multiplier acting on the sign in the multiplicand and of the role of referent 0 are implicit. Moreover, there is rarely any clear intention to build relational understanding between multiplicative objects, such as -50 × -3 = 150 and 50 × 3 = 150, which have the same product through different processes.

Theoretical framework

The Action-Process-Object-Schema (APOS) theory for understanding the construction of knowledge is based on Piaget's reflective abstraction (Dubinsky, 1991). To inform our study of college learners' structures of understanding, we adopted APOS theory as a theoretical framework. An *action* is an experience that a learner has been through without developing a reflective mental construction (e.g., performing the bilinear operation of multiplying signed numbers by following an algorithm). *Process* includes the learner's perceiving the action he or she has performed and being able to reverse or undo the process (e.g., understanding that each part of the bilinear operation of multiplication of signed numbers is necessary and independent of the other). Once a person has reflected on the processes and can perceive it as an entity (e.g., understanding and being flexible with the bilinear nature of operations with signed numbers), then it is said that the process is encapsulated into an *object*. The organized mental structure that includes actions, processes, and objects as well as the ability to use objects and processes as components of some related or more complex action is called a *schema*.

Methodology

The interview protocol for the study of two-factor multiplication was framed in a preparing-for-mathematical-teaching context and was designed to bring to the surface participants' understanding of multiplication. Specifically, given four numerical prompts, (a) 4×3, (b) –4×3, (c) 4×–3, and (d) –4×–3, interviews of three prospective teacher participants followed five steps: (1) computation, (2) problem-posing, (3) visualization of problem-solving, (4) sketch for visualization, and (5) comparison of solutions generated in Steps 1 and 4. All interviews were audio and video recorded and transcribed. Each of the three participants was interviewed one time for approximately ninety minutes. Interview transcripts and tapes were analyzed phenomenologically using constant comparative methods. The participants, who we will refer to pseudonymously as Jenny, Lisa, and Mary, were chosen based on their Math for Elementary Teachers instructor's view that each was excelling in the course at the time and participated in classroom activities with enthusiasm and expertise. After presenting the three parallel cases, we conclude with analysis among the cases.

Results

All three participants had no difficulty when asked to compute the solutions to the multiplication prompts. However, they all had some amount of difficulty in providing story problems for the same prompts. Among the 12 participant-task interview interactions with signed numbers, most resulted in work indicating that the prospective teacher participant did not have a process understanding of negative multiplier. When asked to pose problems that might be used to teach grade school, 6 of 12 participant attempts did not include an appropriate context for negative multiplier and 8 of 12 evidenced that participants could not enact the process-level idea of a multiplier acting on the sign of the multiplicand.

Lisa. Lisa was the only person who posed largely feasible story problems for all the multiplication questions on her first attempts. In Lisa's story problem for 4×3, the multiplier and multiplicand played their usual roles. However, the story problems she had for –4×3 and 4×–3 both regarded the whole number as the multiplier. For 4×–3: "A man who works for the circus owns a monkey who eats three bananas every day. After four days of the monkey eating bananas, how many less bananas will the man have?" Her strategy for creating stories for prompts (b) and (c) was to assign "spending [money]" or "consuming [food]" for the context of negative sign. The answers to the questions that arose in the stories were all in the form of relative status. Lisa's strategy for creating a story about a man walking towards and away from his house for prompt (d) made clear that for her the meaning of the negativeness of the multiplier was operative on the negativeness of the multiplicand. Furthermore, she explicitly pointed out the role of referent 0 in her story:

> Every time there is a negative on the problem, tells him which direction to go. Negative sign on 4 tells him to go to the left but negative on the 3 tells him to turn around and go right from his house, which is at zero on the number line.

Mary. Mary was eventually able to provide story problems for all prompts, though once she referred to a textbook. Like Lisa, Mary regarded the whole numbers in the prompts –4×3 and 4×–3 as the multipliers when posing stories. The only story with context for negative sign she had for 4×–3 was to regard "backwards" as the context for negativeness: "4 people that go backwards 3 steps how [many steps] did they take in which direction?" The question she posed in her story about the steps "they take in which direction" seemed to describe final position as an absolute status without comparing to an original status (i.e., no inherent referent of 0). Despite the reversible process-based affordance of the scenario she offered – negative as walking backwards – Mary seemed not to see the negative in –3 as a process that could be undone. This might be the reason that the prompt –4×–3 was a challenge for her. Mary, when first discussing a context for the negatives in prompt (d), came to a standstill and finally said, "You have negative four groups with negative 3 cars."

For Mary, negative numbers appeared, at best, to be pseudo-objects that could not be de-encapsulated into a value with an associated negative process. Mary was unable to explain what "negative four groups" and "negative 3 cars" could mean. However, after checking with her textbook, Mary's second attempt to pose a problem for –4×–3 was, "If the temperature is now zero degrees, what it was four hours ago if the temperature is decreasing by three degrees every hour?" In terms of Mary's textbook example, she used negative four as multiplier where the quantity was four hours and "ago" was the negative and transformed temperature decreasing to temperature increasing. We are not sure how the concept of multiplier as negative number evolved in Mary's mind when checking her textbook. However, we do know, by her drawing and discussing of her visualization of solving the problem that Mary seemed to grasp, at best procedurally, how to represent negative integer as multiplier.

Jenny. After significant struggle, Jenny gave a story problem for each prompt. Like Lisa and Mary, Jenny regarded the whole numbers in the prompts –4 × 3 and 4 × –3 as the multipliers, despite the difference in placement of the negatives. Jenny's knowledge of signed numbers was clearly impoverished. In explaining her use of negative numbers, Jenny referred to "four red chips" most often and once to "negative four dollars." On her first attempt, she regarded three as the multiplier so that she had three piles of four red chips to represent –4×3. When she was asked to compare the difference between the two expressions, –4×3 and 4×–3, she answered,

> Why I was thinking that is because I chose the positive number because that way you can do -(pause) - since it's going to be negative, and the negative 4

> is represented by the red chips-- That way um, there'd be three piles of four red chips, and then here the positive 1 is the pile [multiplier] again...

Though made up mostly of incompletely expressed thoughts, her answer seemed to confirm that a negative number as multiplier did not have any sense for Jenny.

Discussion

Negative number multiplier. The participants in this study regarded the first factor in multiplication as multiplier only if it was a whole number. If the first factor was not a whole number but the second factor was, they reversed the roles of the two factors. In prompt (d) both of the factors were negative. Mary rarely (and Jenny never) thought of regarding a negative sign on the multiplier in -4×-3 as undoing. The negative sign represented an action for Jenny and the beginnings of a process for Mary. However, their awareness and ownership of "negative" might not have been robust enough to form a reversible process. In other words, Mary and Jenny could only see the static result of action with incomplete mental constructs for the process.

Isomorphic mapping between story context and algorithmic procedure. Lisa was the only one who was able to pose stories and to visualize problem solving for all of the signed number prompts. Her mostly object level of understanding in this matter was evidenced in her ability to map her algorithmic procedures to her visualization and explanation for solving her posed problems. Lisa successfully visualized three essential properties by contextualizing (1) –4 and –3, (2) the operation of –4 on –3, and (3) the product of 12 as a relative position from the referent 0.

Conclusion and implications.

While posing problems, each participant spent some time trying to explain the relationships among contextualized quantities in the problems. Consequently, we conjecture that participants were adjusting their existing understanding for multiplication, folding in the new actions and processes involved in problem-posing as they thought aloud about their efforts to make sense of the request to pose a problem. If a participant was not able to provide a story problem or a context for a purely numeric mathematical object, it likely was an indicator that she failed to retrieve, or manipulate, or modify something she felt was appropriate from memory. Jenny's impoverished understanding of negative numbers and tenacity in keeping a single exemplar (red color) for explaining operations with negatives will certainly impact her teaching if unchanged. Mary recognized the need to refer to the text, and this helped her perform the actions of problem-posing. She also appeared to benefit from the interviewers request to think about the connections between decontextualized computation based on the prompt and what a learner solving her problem would need to do. The participant with the richest understanding, Lisa, was able to pose and make connections to solving her problem for each prompt. We suggest problem-posing, visualization of problem solving, and identifying the isomorphic relationship in between could serve as a tool for investigation as well as for instruction. Further research may need to pay more attention to identifying possible sub-constructs of multiplication schemes from a psychological or developmental perspective, and investigate incorporating these perspectives into curriculum design.

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